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A GENERALIZATION OF DESCARTES' RULE

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A generalization of Descartes' rule*)

by

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ABSTRACT

Let f(z) be an exponential sum

$$f(z) = \sum_{k=1}^{m} c_k \exp(\alpha_k z), \qquad z \in \mathbb{C},$$

where $\alpha_k \in \mathbb{R}$, $c_k \in \mathbb{C}$ $(k = 1,\dots,m)$. We give upper bounds for the number of zeros of f in horizontal strips in the complex plane. By limit transition, these results are extended to exponential integrals

$$f(z) = \int_0^1 e^{zt} g(t)\mu(dt),$$

where g is a continuous complex-valued function and μ a positive measure on [0,1].

KEY WORDS & PHRASES: *Descartes' rule, zeros of functions, exponential polynomials, Laplace transform.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Let f(z) be an exponential sum of the form

$$(1)\qquad f(z) = \sum_{k=1}^{m} c_k \exp(\alpha_k z), \qquad z \in \mathbb{C},$$

where c_k, $\alpha_k \in \mathbb{R}$ $(k = 1,\dots,m)$. We assume this sum to be ordered in such a way that $\alpha_1 < \alpha_2 < \dots < \alpha_n$ and $c_k \neq 0$ $(k = 1,\dots,m)$. Descarte's rule states that the number $N_{\mathbb{R}}(f)$ of zeros of f on the real axis does not exceed the number of sign changes in the sequence $\{c_1,c_2,\dots,c_m\}$ (see POLYA and SZEGÖ [1], chapter 5, problem 77). In this paper we present a generalization of this rule for complex $c_1,\dots,c_m$. We prove

$$(2)\qquad N_{\mathbb{R}}(f) \leq \frac{1}{\pi} \sum_{k=1}^{m-1} |\mathrm{Arg}(c_{k+1}/c_k)|,$$

where $\mathrm{Arg}(\zeta) \in (-\pi,\pi]$ is the principal value of the argument of ζ.

We further use this result to derive an estimate for the imaginary parts of the complex zeros of f. We prove for instance that the number $N_E(f)$ of zeros of f in the strip

$$E = \{z \in C;\ 0 < \mathrm{Im}\, z < r\}$$

satisfies

$$(3)\qquad \pi N_E(f) \leq \sum_{k=1}^{m-1} |\mathrm{Arg}(c_{k+1}/c_k)| + r(\alpha_m-\alpha_1).$$

By limit transition, analogues of (2) and (3) are given for functions of the form

$$(4)\qquad f(z) = \int_0^1 e^{zt}\, g(t)\, \mu(dt),$$

where g(t) is a continuous complex-valued function and μ a positive measure on [0,1], such that f(z) is not identically vanishing. By a simple transformation one can obtain corresponding results for

$$f(z) = \int_a^b e^{zt}\, \mu(dt)$$

where $(a,b) \subset \mathbb{R}$ and μ is a complex measure.

The method by which the above results are derived has been developed by the author in [2]. The method is explained in section 2; results from [2] are briefly sketched and some new ideas are inserted. In section 3 we apply this method to exponential sums (1). Finally, in section 4 we treat the exponential integral (4).

2. TOTAL VARIATION OF THE ARGUMENT OF A FUNCTION

Let $f(z)$ be an analytic complex-valued function in an open neighbourhood $\mathcal{O}$ of the interval $L = [a,b] \subset \mathbb{R}$. On L, $\mathrm{Im}(f'/f)$ is the restriction of an analytic function (see [2], Lemma 1). If $f(a)f(b) \neq 0$ we define

$$(5) \qquad A(a,b;f) = \int_a^b |\mathrm{Im}(f'(t)/f(t))|\,dt + \pi\, N_L(f),$$

where $N_L(f)$ is the number of zeros of f in L. Here and everywhere else in this paper multiple zeros are counted according to their multiplicities. If $\arg f(t)$, the argument of f, is suitably defined, $A(a,b;f)$ appears to be the total variation of $\arg f(t)$. Since f is analytic in $\mathcal{O}$, there is a $\theta > 0$ such that for $0 < \varepsilon < \theta$ the functions

$$f_\varepsilon(x) := f(x+i\varepsilon)$$

are analytic on L. The following properties of $A(a,b;f)$ hold.

LEMMA 1. *If* $f(a)f(b) \neq 0$ *then*

$$\lim_{\varepsilon\to 0} A(a,b;f_\varepsilon) = A(a,b;f).$$

PROOF. See [2], Lemma 2.

LEMMA 2. *Let* $f_1(z), f_2(z), \ldots$ *be a sequence of complex-valued functions, analytic and uniformly converging to* $f(z)$ *in* $\mathcal{O}$. *If* $f(a)f(b) \neq 0$, *then*

$$\lim_{k\to\infty} A(a,b;f_k) = A(a,b;f).$$

PROOF. Let K be a compact subset of $\mathcal{O}$, containing an open neighbourhood $\mathcal{O}'$ of [a,b]. Since f(z) is analytic in $\mathcal{O}$, the number of zeros of f in K is finite. Hence there is a $\theta > 0$ such that for $0 < \varepsilon < \theta$ the functions $f_\varepsilon(x)$ have no zeros on [a,b]. For $0 < \varepsilon < \theta$ the functions $f'_k(x+i\varepsilon)$ converge to $f'_\varepsilon(x)$ and, since $f_\varepsilon(x) \neq 0$, the functions $1/f_k(x+i\varepsilon)$ converge to $1/f_\varepsilon(x)$ uniformly on [a,b]. Thus for $0 < \varepsilon < \theta$

$$\lim_{k\to\infty} A(a,b;f_k(x+i\varepsilon)) = A(a,b;f_\varepsilon).$$

By letting $\varepsilon \downarrow 0$ and applying Lemma 1, the lemma is proved.

LEMMA 3. *Let* f, g *be analytic in* $\mathcal{O}$. *If* $f(a)f(b)g(a)g(b) \neq 0$, *then*

$$A(a,b;fg) \leq A(a,b;f) + A(a,b;g).$$

PROOF. Straightforward (see [2], Lemma 3).

For complex $z \neq 0$, we define Arg(z), the principal value of the argument of z, in such a way that $-\pi < \text{Arg } z \leq \pi$. Observe that the function $|\text{Arg }(z)|$ is continuous for all $z \in \mathbb{C}$, $z \neq 0$. We now prove our main theorem.

THEOREM 1. *Let* f *be analytic in* $\mathcal{O}$. *If* $f(a)f'(a)f(b)f'(b) \neq 0$, *then*

$$A(a,b;f) \leq A(a,b;f') + \psi(a) - \psi(b),$$

where

$$\psi(x) = |\text{Arg}(f'(x)/f(x))|. \tag{6}$$

PROOF. See [2], Theorem 1, but omit the estimate $\psi(b) - \psi(a) \geq -\pi$. The only new element needed is the observation that

$$\lim_{\varepsilon\to 0} \psi(a+i\varepsilon) = \psi(a); \quad \lim_{\varepsilon\to 0} \psi(b+i\varepsilon) = \psi(b),$$

which follows from $f(a)f(b)f'(a)f'(b) \neq 0$.

3. EXPONENTIAL SUMS

In this section we consider functions

$$f(z) = \sum_{k=1}^{m} c_k \exp(\alpha_k z), \tag{7}$$

where $\alpha_k \in \mathbb{R}$, $0 \neq c_k \in \mathbb{C}$ $(k = 1,\dots,m)$. Without loss of generality we will assume in this section that $\alpha_1 < \alpha_2 < \dots < \alpha_m$. We prove the following result.

THEOREM 2. *Let* $f(z)$ *be given by* (7). *For all* $a,b \in \mathbb{R}$ *such that* $f(a)f(b) \neq 0$

$$A(a,b;f) \leq \sum_{k=1}^{m-1} |\mathrm{Arg}(c_{k+1}/c_k)|.$$

PROOF. The theorem is proved by induction on m. If $m = 1$ the theorem is trivial, since $A(a,b;c_1 \exp(\alpha_1 x)) = 0$. Suppose that $m > 1$. By lemma 3, $A(a,b;f(x)\exp(-\alpha_1 x)) = A(a,b;f(x))$. We may thus assume without loss of generality that $\alpha_1 = 0$. Now we can choose arbitrarily large numbers $t \in \mathbb{R}$ such that $f(t)f'(t)f(-t)f'(-t) \neq 0$. By Theorem 1 we have

$$A(a,b;f) \leq \lim_{t\to\infty} A(-t,t;f) \leq \lim_{t\to\infty}(A(-t,t;f')+\psi(-t)+\psi(t)). \tag{8}$$

Here $\psi(x) = |\mathrm{Arg}(f'(x)/f(x)|$, according to (6). Since $\alpha_1 = 0$,

$$f'(x) = \sum_{k=2}^{m} c_k \alpha_k \exp(\alpha_k z).$$

Hence

$$\lim_{t\to\infty} \psi(t) = \lim_{t\to\infty} \left|\mathrm{Arg}\left(\frac{c_m\alpha_m e^{\alpha_m t}(1+\theta_1(t))}{c_m e^{\alpha_m t}(1+\theta_2(t))}\right)\right|$$

where $\theta_1(t)$ and $\theta_2(t)$ tend to zero if $t \to \infty$. So $\lim_{t\to\infty} \psi(t) = 0$. Similarly

$$\lim_{t\to\infty} \psi(-t) = \lim_{t\to\infty} \left|\mathrm{Arg}\left(\frac{c_2\alpha_2 e^{-\alpha_2 t}(1+\theta_3(t))}{c_1(1+\theta_4(t))}\right)\right| = |\mathrm{Arg}(c_2/c_1)|.$$

So (8) becomes

$$A(a,b;f) \leq \lim_{t\to\infty} A(-t,t;f') + |\mathrm{Arg}(c_2/c_1)|. \tag{9}$$

By the induction hypothesis we have

$$A(-t,t;f') \leq \sum_{k=2}^{m-1} |\mathrm{Arg}(\alpha_{k+1}c_{k+1}/\alpha_k c_k)| = \sum_{k=2}^{m-1} |\mathrm{Arg}(c_{k+1}/c_k)|, \tag{10}$$

since the exponential sum f' has only $m-1$ terms. On inserting (10) in (9) the desired result is obtained.

Since $\pi N_{\mathbb{R}}(f) \le \lim_{t\to\infty} A(-t,t;f)$, Theorem 2 yields (2). This shows that Theorem 2 is a pure generalization of Descartes' rule. The next result combines Theorem 2 with a method of WILDER [3].

THEOREM 3. *Let* $f(z)$ *be given by* (7). *The number* $N_E(f)$ *of zeros of* f *in the strip*

$$E = \{z \in \mathbb{C};\ r \le \operatorname{Im} z \le s\}$$

satisfies

$$(11)\qquad |2\pi N_E(f)-(s-r)(\alpha_m-\alpha_1)| \le \sum_{k=1}^{m-1} \left|\left(\frac{c_{k+1}\exp(ir\alpha_{k+1})}{c_k\exp(ir\alpha_k)}\right)\right| + \left|\operatorname{Arg}\left(\frac{c_{k+1}\exp(is\alpha_{k+1})}{c_k\exp(is\alpha_k)}\right)\right|.$$

PROOF. Choose a number ε with $0 < \varepsilon < 1$. There is a $T > 0$ such that

$$(12)\qquad \begin{aligned} f(x+iy) &= c_m \exp(\alpha_m(x+iy))(1+\theta_1(x,y)) \\ f(-x+iy) &= c_1 \exp(\alpha_1(-x+iy))(1+\theta_2(x,y)) \\ f'(x+iy) &= c_m\alpha_m \exp(\alpha_m(x+iy))(1+\theta_3(x,y)) \\ f'(-x+iy) &= c_1\alpha_1 \exp(\alpha_1(-x+iy))(1+\theta_4(x,y)), \end{aligned}$$

where $|\theta_i(x,y)| < \varepsilon$ for $r \le y \le s$ and $x \ge T$ $(i = 1,\dots,4)$. Since $\varepsilon < 1$, we find by Rouché's theorem that all zeros of f in E lie in the smaller set

$$E_1 = \{z \in \mathbb{C};\ r \le \operatorname{Im} z \le s, |\operatorname{Re} z| \le T\}.$$

Denote by δE_1 the (positively oriented) boundary of E_1. Under the assumption that there are no zeros of f on the boundary of E and thus on δE_1, we have

$$2\pi N_E(f) = 2\pi N_{E_1}(f) = -i\int_{\delta E_1}\left(\frac{f'(z)}{f(z)}\right)dz = \operatorname{Im}\left(\int_{\delta E_1}\frac{f'(z)}{f(z)}\,dz\right) =$$

$$= \operatorname{Im}\int_{-T}^{T}\left(\frac{f'(x+ir)}{f(x+ir)} - \frac{f'(x+is)}{f(x+is)}\right)dx + \operatorname{Re}\left(\int_r^s\left(\frac{f'(T+iy)}{f(T+iy)} - \frac{f'(-T+iy)}{f(-T+iy)}\right)dy\right).$$

By (12)

$$\mathrm{Re}\left(\int_r^s \frac{f'(T+iy)}{f(T+iy)}\,dy\right) = \mathrm{Re}\left(\int_r^s \alpha_m\left(\frac{1+\theta_1(T,y)}{1+\theta_2(T,y)}\right)dy\right) = (s-r)\alpha_m(1+\theta_5),$$

where $|\theta_5| \leq \frac{1+\varepsilon}{1-\varepsilon} - 1 = \frac{2\varepsilon}{1-\varepsilon}$. Similarly

$$\mathrm{Re}\left(\int_r^s \frac{f'(-T+iy)}{f(-T+iy)}\,dy\right) = (s-r)\alpha_1(1+\theta_6),$$

where $|\theta_6| \leq 2\varepsilon/(1-\varepsilon)$. So we find after some simple estimations

(13) $$|2\pi N_E(f)-(s-r)(\alpha_m-\alpha_1)| \leq A(-T,T;f(x+ir))+A(-T,T;f(x+is))+\theta_7,$$

where $|\theta_7| \leq 4\varepsilon/(1-\varepsilon)$. Now

$$f(x+ir) = \textstyle\sum_{k=1}^m c_k \exp(ir\alpha_k) \exp(\alpha_k x)$$

$$f(x+is) = \textstyle\sum_{k=1}^m c_k \exp(is\alpha_k) \exp(\alpha_k x)$$

and a straigthforward application of Theorem 2 transforms (13) into

(14) $$|2\pi N_E(f) - (s-r)(\alpha_m-\alpha_1)| \leq$$

$$\leq \textstyle\sum_{k=1}^{m-1} \left|\mathrm{Arg}\left(\frac{c_{k+1}\exp(ir\alpha_{k+1})}{c_k\exp(ir\alpha_k)}\right)\right| + \sum_{k=1}^{m-1} \left|\mathrm{Arg}\left(\frac{c_{k+1}\exp(is\alpha_{k+1})}{c_k\exp(is\alpha_k)}\right)\right| + \theta_7.$$

Letting $\varepsilon \downarrow 0$ the theorem is proved under the assumption that no zeros of f lie on the boundary of E. We now drop this assumption. There is a $t > 0$ such that for $0 < \rho < t$ the strips

$$E(\rho) := \{z \in \mathbb{C};\ r-\rho \leq \mathrm{Im}\ z \leq s+\rho\}$$

have no zeros on their boundaries. We thus find that $N_E(f) = N_{E(\rho)}(f)$ for $0 < \rho < t$, whereas

$$|2\pi N_{E(\rho)}(f) - (s-r+2\rho)(\alpha_m-\alpha_1)| \le$$

$$\le \sum_{k=1}^{m-1} \left|\mathrm{Arg}\left(\frac{c_{k+1}\exp(i(r-\rho)\alpha_{k+1})}{c_k\exp(i(r-\rho)\alpha_k)}\right)\right| + \left|\mathrm{Arg}\left(\frac{c_{k+1}\exp(i(s+\rho)\alpha_{k+1})}{c_k\exp(i(s+\rho)\alpha_k)}\right)\right|.$$

By letting $\rho \downarrow 0$ and observing that in the above inequality both terms are continuous functions of ρ the theorem is proved completely.

The following lemma is useful for the evaluation of (11).

LEMMA 4. *For complex* $y,z \neq 0$ *we have*

$$|\mathrm{Arg}(yz) \le |\mathrm{Arg}(y)| + |\mathrm{Arg}(z)| \le 2\pi - |\mathrm{Arg}(y/z)|.$$

PROOF. Since $\mathrm{Arg}(\zeta) \in (-\pi,\pi]$, we have either

$$\mathrm{Arg}(yz) = \mathrm{Arg}(y)+\mathrm{Arg}(z)$$

or

$$|\mathrm{Arg}(y) + \mathrm{Arg}(z)| > \pi.$$

In both cases we trivially have

$$|\mathrm{Arg}(yz)| \le |\mathrm{Arg}(y)| + |\mathrm{Arg}(z)|.$$

The following relations are easily checked

$$|\mathrm{Arg}(-\zeta)| = \pi - |\mathrm{Arg}(\zeta)|; \ |\mathrm{Arg}(1/\zeta)| = |\mathrm{Arg}(\zeta)|.$$

Hence

$$|\mathrm{Arg}(y)| + |\mathrm{Arg}(z)| = 2\pi-(|\mathrm{Arg}(-y)| + |\mathrm{Arg}(-1/z)|) \le 2\pi - |\mathrm{Arg}(y/z)|.$$

From Theorem 3 we now derive

COROLLARY 1. *Let* f *and* E *be as in Theorem* 3. *If* $r < 0 < s$, *then*

$$\pi N_E(f) \le \sum_{k=1}^{m} |\mathrm{Arg}(c_{k+1}/c_k)| + (s-r)(\alpha_m-\alpha_1).$$

PROOF. We have by Lemma 4

$$\left|\operatorname{Arg}\left(\frac{c_{k+1}\exp(ir\alpha_{k+1})}{c_k\exp(ir\alpha_k)}\right)\right| \le \left|\operatorname{Arg}\left(\frac{c_{k+1}}{c_k}\right)\right| + \left|\operatorname{Arg}(\exp(ir(\alpha_{k+1}-\alpha_k)))\right| \le$$

$$\le \left|\operatorname{Arg}\left(\frac{c_{k+1}}{c_k}\right)\right| + |r|(\alpha_{k+1}-\alpha_k) = \left|\operatorname{Arg}\left(\frac{c_{k+1}}{c_k}\right)\right| - r(\alpha_{k+1}-\alpha_k).$$

By inserting these and similar estimates in (11) the corollary follows.

COROLLARY 2. *Let* f, E *be as in Theorem* 3. *Then*

$$|2\pi N_E(f)-(s-r)(\alpha_m-\alpha_1)| \le 2\pi(m-1) - \sum_{k=1}^{m-1} |\operatorname{Arg}(\exp(i(s-r)(\alpha_{k+1}-\alpha_k)))|.$$

PROOF. By Lemma 4

$$\left|\operatorname{Arg}\left(\frac{c_{k+1}\exp(ir\alpha_{k+1})}{c_k\exp(ir\alpha_k)}\right)\right| + \left|\operatorname{Arg}\left(\frac{c_{k+1}\exp(is\alpha_{k+1})}{c_k\exp(is\alpha_k)}\right)\right| \le$$

$$\le 2\pi - |\operatorname{Arg}(\exp(i(s-r)(\alpha_{k+1}-\alpha_k)))|$$

giving the desired inequality by insertion in (11).

COROLLARY 3. *Let* f, E *be as in Theorem* 3. *If* $(s-r)(\alpha_{k+1}-\alpha_k) < \pi$ *for* $k = 1,\dots,m-1$, *then*

$$N_E(f) \le m-1.$$

PROOF. By Corollary 2 and the fact that $\operatorname{Arg}(\exp(i(s-r)(\alpha_{k+1}-\alpha_k))) = (s-r)(\alpha_{k+1}-\alpha_k)$ we find

$$|2\pi N_E(f)-(s-r)(\alpha_m-\alpha_1)| \le 2\pi(m-1)- \sum_{k=1}^{m-1} (s-r)(\alpha_{k+1}-\alpha_k).$$

Hence,

$$2\,N_E(f)-(s-r)(\alpha_m-\alpha_1) \le 2\pi(m-1)-(s-r)(\alpha_m-\alpha_1).$$

4. EXPONENTIAL INTEGRALS

Let g be a not identically vanishing continuous complex function on the interval $I=[0,1]$ and let μ be a positive measure on I. We consider not identically vanishing functions of the form

$$(15) \qquad f(z) = \int_0^1 e^{zt} g(t)\mu(dt).$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ with $0 = x_0 < x_1 < \ldots < x_n = 1$ be a partition of I and put $I_k := [x_{k-1}, x_k)$ for $k = 1, \ldots, n-1$; $I_n := [x_{n-1}, x_n]$. We now construct the exponential sums

$$(16) \qquad F_P(z) = \sum_{k=1}^n e^{x_k z} g(x_k)\mu(I_k)$$

and observe that

$$|F_P(z) - f(z)| = |\sum_{k=1}^n \int_{I_k} (e^{x_k z} g(x_k) - e^{tz} g(t))\mu(dt)| \le$$

$$\le \mu(I) \max_{k=1,\ldots,n} \max_{t \in I_k} |e^{x_k z} g(x_k) - e^{tz} g(t)|.$$

Since $g(t)e^{zt}$ is uniformly continuous on I, the following lemma is immediately proved by letting $\max_k |x_k - x_{k-1}| \to 0$.

<u>LEMMA 5</u>. *Let* $D \subset \mathbb{C}$ *be bounded. Then there is a sequence* $P_1, P_2, \ldots$ *of partitions of* I *such that* $\lim_{k\to\infty} (F_{P_k}(z)) = f(z)$ *uniformly for* $z \in D$.

We thus have by Lemma 2 that if $(a,b) \subset \mathbb{R}$ such that $f(a)f(b) \neq 0$

$$A(a,b;f) \le \sup_P A(a,b;F_P).$$

Let the sequence $y_1, \ldots, y_m$ be derived from $x_1, \ldots, x_n$ by deleting those x_k for which $g(x_k)\mu(I_k) = 0$. Then by Theorem 2 if P is chosen such that $\{y_1, \ldots, y_m\} \neq \emptyset$,

$$A(a,b;F_P) \le \sum_{k=1}^{m-1} |\mathrm{Arg}(g(y_{k+1})/g(y_k))|.$$

Now define

$$A^*(g) = \sup_P \sum_{k=1}^{m-1} |\mathrm{Arg}(g(y_{k+1})/g(y_k))|,$$

the supremum being taken over all partitions P. If arg(g), the argument of g, is defined as a right-continuous function, left continuous whenever $g(x) \neq 0$ and such that $\lim_{\varepsilon \downarrow 0} |\arg(g(x+\varepsilon)) - \arg(g(x-\varepsilon))| \leq \pi$ whenever $g(x) = 0$, then $A^*(g)$ is the total variation of $\arg(g(x))$ on $[0,1]$.

As an immediate consequence of Lemma 2, Theorem 2 and Lemma 5 we have the following theorem

THEOREM 4. *Let* $f(z)$ *be given by* (15). *For all* $a < b \in \mathbb{R}$ *such that* $f(a)f(b) \neq 0$

$$A(a,b;f) \leq A^*(g).$$

PROOF. Straigthforward.

COROLLARY 4. *The number* $N_{\mathbb{R}}(f)$ *of real zeros of* f *satisfies*

$$N_{\mathbb{R}}(f) \leq \frac{1}{\pi} A^*(g).$$

In order to estimate the number of zeros of f in a horizontal strip, we derive the following lemma, somewhat similar to Lemma 2.

LEMMA 6. *Let* $f_1, f_2, \ldots$ *be a sequence of analytic complex-valued functions uniformly converging to* f *in a compact set* $K \subset \mathbb{C}$. *If* f *has no zeros on the boundary of* K, *then there is an* $N > 0$ *such that for* $k > N$ *the functions* f *and* f_k *have the same number of zeros in* K.

PROOF. Denote the boundary of K by δK. There is an $\varepsilon > 0$ such that $|f(z)| > \varepsilon$ for $z \in \delta K$. There is an $N > 0$ such that for $k > N$ and $z \in \delta K$

$$|f_k(z) - f(z)| < \varepsilon.$$

So by Rouché's theorem, $f_k(z)$ and $f(z)$ have the same number of zeros in K.

We now derive the following theorem

THEOREM 5. *Let* $f(z)$ *be given by* (15) *and let* $r, s \in \mathbb{R}$ *such that* $r < 0 < s$. *Then the number* $N_E(f)$ *of zeros of* f *in the strip*

$$E = \{z \in \mathbb{C};\ r \leq \operatorname{Im} z \leq s\}$$

satisfies

$$\pi N_E(f) \le A^*(g) + s-r.$$

<u>PROOF</u>. Suppose that f has no zeros on the boundary of E. Choose T > 0 such that f has no zeros on the boundary of E_1, where

$$E_1 = \{z \in C;\ r \le \text{Im } z \le s;\ |\text{Re}(z)| \le T\}.$$

By Lemma's 5 and 6

$$N_{E_1}(f) = \lim_{k\to\infty} N_{E_1}(F_{P_k}).$$

By Corollary 1, since $y_m - y_1 \le 1$ for all partitions P

$$N_{E_1}(F_{P_k}) \le N_E(F_{P_k}) \le A^*(g) + s-r.$$

By letting $T \to \infty$ the theorem follows. If f has zeros on the boundary of E, there is a $\delta_1 > 0$ such that for $0 < \delta < \delta_1$ f has no zeros on the boundary of E_δ, where

$$E_\delta = \{z \in \mathbb{C};\ r-\delta \le \text{Im } z \le s+\delta\}.$$

Now $N_E(f) \le N_{E_\delta}(f) \le A^*(g) + s-r+2\delta$.

By letting $\delta \downarrow 0$ the theorem is proved completely.

5. REMARKS

1. Let $a,b \in \mathbb{C}$, Γ the rectilinear segment running from a to b, μ a positive measure and g a complex-valued continuous functions on Γ. If

$$f(z) = \int_\Gamma g(\zeta)e^{z\zeta}\mu(d\zeta),$$

then

$$f(z) = e^{az}\int_0^1 g(a+(b-a)t)e^{(b-a)tz}\mu^*(dt) = e^{az}h((b-a)z).$$

Here μ^* is a positive measure on [0,1] derived from μ. We can apply the results of section 4 to h and so derive corresponding results for f.

2. The representation (15) is not unique; if h is a positive μ-measurable function on [0,1],

$$\int_0^1 e^{zt} g(t)\mu(dt) = \int_0^1 e^{zt}(g(t)/h(t))\mu^*(dt),$$

where μ^*, defined by $\mu^*([\alpha,\beta]) = \int_\alpha^\beta h(t)\mu(dt)$, is again a positive measure. We can thus somewhat alleviate the condition that g is continuous, by letting $h \to \infty$ at the points of discontinuity of g, h continuous everywhere else.

3. The method in this paper is not limited to exponential sums and integrals; for instance Dirichlet series

$$\sum_{k=1}^{\infty} c_k \exp(-\lambda_k x)$$

and infinite integrals

$$\int_0^\infty g(t)\, e^{-zt} dt$$

can be represented as limits of exponential sums. Proceeding as in section 4, analogues of Theorem 4 can be derived for these functions. Compare PÓLYA & SZEGÖ [1], Chapter 5, section 1, especially problems 78 and 80.

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